Examining the Correlation Between Imaginary Numbers and Reality Through Their Physical Applications

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Abstract

Theorems in physics have increasingly become disconnected with reality. Imaginary numbers now appear in applications such as electrodynamics and quantum mechanics. For example, imaginary numbers appear in the electromagnetic wave function and the Schrödinger equation. Since both of these applications describe a physical system, it is not clear why imaginary numbers are involved in the mathematics. This study will examine these applications and their derivations, in order to determine a distinct connection to the physical world. If the theorems analyzed in the paper describe physical phenomena and rely on imaginary numbers to work, then they are considered necessary for the theorem. If they are not necessary for the concept, they will be considered a mathematical convenience that has no connection to reality. After analyzing each application listed in the methods section, this research paper concluded that imaginary numbers are a mathematical convenience for the electromagnetic wave function, but they do prove to be necessary in the Schrödinger equation. However, since the Schrödinger equation and its solution are extremely vague in their physical relevance, the imaginary number's connection to reality is quite vague as well. This is because the imaginary number is only used in the process to achieve the physical connection, not in the physical connection itself. Therefore, this study determines that the physical relevance of the solution to the Schrödinger equation and the Schrödinger equation itself must be understood clearly, in order to find a distance connection between the imaginary number and reality. In addition, this paper also demonstrates that the "vagueness" of Schrödinger's equation and solution and the fact that it depends on the imaginary number may indicate flawed aspects behind quantum mechanical theories.

Examining the Correlation Between Imaginary Numbers and Reality Through Their Physical Applications

Mathematics is a science that generates mathematical constructs to represent physical phenomena. It uses these mathematical constructs to generalize concepts about objective reality. The Babylonians, Greeks, and most mathematicians before the tenth century viewed mathematics with this perspective. Mathematics was steadily developed on premises that held true to objective reality, and the motivations for the proof were from observations. After this period, mathematicians have started to motivate their findings a lot less. Using a more deductive approach, mathematicians make fundamentally correct theorems but do not know their implications to reality because they were deduced from other theories and were not motivated by physical observations. Today, the mathematical field of topology represents this approach. This approach develops many equations through mathematical deductions, which are not wrong, but are also not explicitly motivated by physical observations. As a result, the relations between these abstractions had an unclear physical relevance. Eventually, mathematicians and physicists started describing reality as subjective in the 19th and 20th centuries, which eliminated any incentive to establish a connection to reality. Although many of these mathematicians most likely understood the deeper connections of their own ideas to reality, they did not care to communicate these findings because it did not matter: reality was subjective. As a result, many students describe mathematics as a difficult field, since they cannot see what led mathematicians to create these theorems. A clear example of this phenomenon is the use of imaginary numbers within physical applications.

Although this investigation relies on mathematical and physical concepts, the primary thesis is epistemological; it relies on the way in which concepts should relate to reality. Multivariable calculus, differential equations, classical mechanics, and linear algebra are used throughout the paper. Although there are many concepts from quantum mechanics and electrodynamics within the paper, they will be explained to a satisfactory extent for someone who has the earlier mentioned prerequisites.

The imaginary number is defined as

1.1 $$i \equiv \sqrt{-1} \quad \therefore \quad i^2 = -1$$

It can be joined with real numbers to represent a complex number.

1.2 $$z = a + bi \quad a, b \in \mathbb{R}$$

Complex numbers have the following properties

1.3 $$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$

1.4 $$(a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i$$

1.5 $$(a_1 + b_1 i) \cdot (a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$$

1.6 $$\frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{a_1 + b_1 i}{a_2 + b_2 i}\frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2)i}{a_2^2 + b_2^2}$$

1.7 $$\text{RE}(z) = a \quad \text{IM}(z) = b$$

1.8 $$|z| = \sqrt{a^2 + b^2}$$

Similar to real quantities, complex numbers are defined as a number system, where complex numbers represent a vector; equations 1.3 through 1.6 define the properties of this number system. Complex numbers were controversial when they were first “discovered”. This is because there is no physical quantity that, when squared, produces a negative number. However, they gradually became incorporated into more theorems. One of the first applications of complex numbers, the fundamental theorem of algebra, states that a polynomial has as many roots (which may be complex) as its order. It was created by the seventeenth century mathematician, Descartes. The complex numbers do not have physical relevance here because this is not a physical application; the theorem (and the Descartes rule of sign) only allows for one to mathematically reason the possibilities for a solution for any given polynomial. However, without complex numbers, Descartes's theorem would be invalid. This is because certain polynomials do not have as many real solutions as their order; however, they do have as many solutions if complex solutions are considered.

The acceptance of complex numbers followed a similar path to that of zero and negative numbers. Early mathematicians believed "nothing" (zero) was not a valid quantity and that there cannot be a "negative" amount of something. However, early mathematicians slowly realized that these concepts describe the physical world. For example, one can have a negative amount of money, known as debt; it is also beneficial to have a concept of zero when balancing budgets or when two quantities cancel each other. Negative numbers are related to positive numbers in a way such that they cancel each other, such as opposing directions. The problem with imaginary numbers is that they do not relate to reality in the same way. There is no physical quantity that when multiplied by itself, produces a negative quantity. Despite this, imaginary numbers were adopted by later mathematicians and physicists. For example, Euler, an eighteenth century mathematician, created a new field of study called complex analysis where he defined the following deductive relation

1.9 $$e^{i\theta} = \cos\theta + i\sin\theta$$

by manipulating the following Taylor series.

1.10 $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

1.11 $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

1.12 $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

The following relations are then found to be equal.

1.13 $$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + i\frac{x^7}{7!} + \ldots$$

1.14 $$\cos x + i \sin x = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + i\frac{x^7}{7!} + \ldots$$

After this realization, mathematicians constantly created new formulas using Euler's formula as a premise. However, it was not until electrodynamics and quantum mechanics that imaginary numbers were used in equations representing the nature of reality. This investigation aims to answer the question of "How it is possible for quantities that are not real to be used in physical applications?"

Earlier mathematicians proposed a similar question to that of this study. However, this was before imaginary numbers were involved in physical applications and they often misinterpreted reality when they tried to answer the question. Caspar Wessel, a Danish mathematician, attempted to relate imaginary numbers to real numbers graphically. He invented the complex plane to accomplish such task. The complex plane is a cartesian coordinate system where real numbers are graphed on the x-axis and imaginary numbers are graphed on the y-axis.

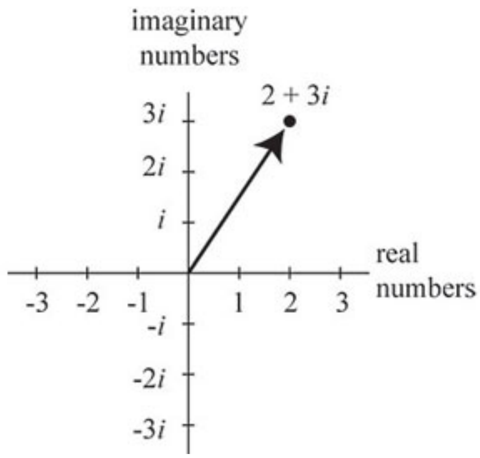

Figure 1 Image of complex plane (Tapp, D. 2014)

Wessel could now plot real numbers against imaginary numbers. However, this does not relate imaginary numbers to real numbers. He assumed that a relationship already existed because the complex field was created on the premise that real numbers and imaginary numbers can be related in such a way.

Another invalid argument is that imaginary numbers correlate with real numbers by a rotation factor of $\pi/2$ radians. However, this only works in the complex field.

Theorems following the twentieth century treat reality as subjective; therefore, some physicists fail to see the validity of the question. The question this paper asks has never been posed because it has a different philosophical framework: concepts must form a grasp of a relationship found in reality for that concept to be validated, whereas, most people who work in this field fail to see this principle.

Hypothesis

Imaginary numbers present themselves in physical applications. Since the imaginary number is defined to be unable to represent a physical quantity, the study suspects three possible outcomes of this investigation:

1. Imaginary numbers provide a mathematical convenience to simplify the equation algebraically.
2. Although the theorem may accurately provide information for the system, the underlying concept of the theorem may be flawed or incomplete.
3. The full definition of imaginary numbers are not completely understood because they do have some physical existence.

Notice that each outcome is independent of one another and 1, 2, and 3 predict all of the possible outcomes. The goal of this paper is to analyze some concepts that imaginary numbers are involved in to determine which of these outcomes is true.

It is important to include a 4th outcome that includes both the second and third outcomes previously stated. This fourth outcome has the philosophical belief that reality is subjective. It states that imaginary numbers do not have physical relevance because nothing does. The theorem cannot be validated because there is no concept of validity; the theorem's purpose is to describe a set of observables accurately, not the fundamental nature of reality. This fourth outcome is usually said in conjunction to the phrase: the question on the physical relevance of the imaginary number should not be posed. Although it is unlikely for imaginary numbers to have some physical context, the question should still be posed; not posing the question would be turning a blind eye to the scientific method. Without an answer to the question, the theorem cannot be validated.

The following philosophical perspective was first created when Ayn Rand founded Objectivism in the twentieth century. The view that concepts cannot be validated is a violation of multiple metaphysical axioms. Things that exist follow the law of identity; they have certain properties that make them distinct from others. As a result, they interact with other things that exist in distinct ways, according to the properties the things possess, which is known as the law of causality. Human senses receive these properties a specific way, because each thing can only affect other things in specific ways, unique to itself. Although the senses can be misinterpreted by people, they are not wrong; as a result, concepts can be validated. This is because these concepts describe properties of existing things that have unique properties and the concepts that are created about these properties are either right, wrong, or incomplete. Therefore, the 4th outcome should not be considered an outcome and the question posed in this research still remains valid.

Methods

In order to achieve an answer to whether or not imaginary numbers have a connection with reality, the imaginary number's physical applications must be analyzed. In this case, the term "analyzed" is used to mean the following: tracking the significance of the imaginary numbers in the derivation for each physical relation to determine if they are necessary to the relation. As a result, this research paper will analyze several of the imaginary number's applications within electrodynamics and quantum mechanics. The derivation of the electromagnetic wave equation

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} \quad \tilde{A} \equiv Ae^{i\delta}$$

will be analyzed; within quantum mechanics, the derivation for the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

the wave function itself, Born's interpretation of wave functions,

$$\int |\psi|^2$$

the solution to the Schrödinger equation for the hydrogen atom,

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{\frac{-r}{na}} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\right] Y_l^m(\theta, \phi)$$

$$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos\theta)$$

and Bohm's pilot wave theory

$$\frac{dX}{dt} = \frac{j}{\rho}\bigg|_{x=X(t)}, \quad j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right), \quad \rho = |\psi|^2$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

will be analyzed.

Notice how the equations above describe physical systems but also contain imaginary numbers. If the analysis of these particular applications proves that the imaginary number is necessary for any one of these theorems, then the following statement will be valid: imaginary numbers do have a connection to reality. This statement would be considered valid because imaginary numbers would be vital in the proof of the theorem, which does correspond to reality.

Analysis of Electrodynamics

Imaginary numbers appear in electromagnetic wave functions in the form of

2.1 $$\tilde{f}(z,t) \equiv \tilde{A} e^{i(kz-\omega t)} \quad \tilde{A} \equiv A e^{i\delta}$$

The function depends on the imaginary number. This equation only has two independent variables z and t; however, the electromagnetic wave function will usually be seen with the independent variables x, y, z, and t. This research paper uses the one dimensional function because it can be considered the most general and simple. It is unnecessary to prove the same conclusion with a more difficult, specific version of the function when the same conclusion can be reached with an easier, general version of the function. Therefore, the derivation of this function will be analyzed to reach one of the outcomes listed in the hypothesis.

Before the derivation is introduced, a brief summary of the motivation for the derivation will be shown. James Maxwell, a nineteenth century physicist, arrived at the realization that electromagnetic functions were waves by manipulating his free space formulas

2.2 $$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

to prove the following

2.3 $$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

which is a differential equation whose solution is a wave. As a result, he determined his equations, which modeled electromagnetic functions, produced a differential equation in which the solutions to the electric field and magnetic fields were waves. Maxwell then concluded that electromagnetic functions in free space were waves.

The following derivation is based off of Dr. Griffiths's derivation, a physicist from Harvard University, although investigation provides different explanations and additional steps [3].

Since there is notable evidence to consider the electromagnetic function as a wave, there is significant motivation to define a function resembling these properties. For the more general, one dimensional function, imagine a string that was displaced from equilibrium and a wave traveling along the string. In other words, the physical example used for modeling purposes is an infinite length string that was yanked up and down like a whip. The disturbance in the string (the wave) will propagate in one direction and will change as a function of time. Statement 2.4 defines the following relation.

2.4 $$f = f(z, t)$$

where z represents the direction that the wave function is propagating in and t represents time. The initial condition for time is defined as

2.5 $$g(z_0) \equiv f(z_0, 0)$$

and

2.6 $$g(z) \equiv f(z, 0)$$

where z_0 represents the starting point of the function. For now, this arbitrary value is considered the relative maximum of the wave of the first "wave cycle," in order to visualize the function. The length between the waves that are produced is represented by vt, where v is velocity. This makes intuitive sense as this represents a change in distance after time has passed. The initial condition can then be rewritten as

2.7 $$f(z, t) = f(z_1 - vt, 0) = g(z_1 - vt)$$

where

2.8 $$z_1 = z_0 + vt$$

In other words, z_1 represents the next relative maximum at the same height as the first wave since the function repeats. Equation 2.7 can be further generalized.

2.9 $$f(z,t) = f(z - vt, 0) = g(z - vt)$$

This is the defining property of the electromagnetic wave. Another fundamental property of the function would be the following

2.10 $$\Delta F = T\sin\theta' - T\sin\theta$$

which models the vertical force at any given point on the infinite string. This equation comes from Newton's second law and can be fairly intuitive as well. θ is at the point z and θ' is at the point $z + \Delta z$ where Δz is infinitely small. θ and θ' represent the angles formed from the horizontal axis and the tension unit vector. The two vertical forces being subtracted causes a change in the net force in one direction. Since θ and θ' are infinitely small angles, the following identity follows

2.11 $$\sin\theta = \tan\theta$$

Equation 2.10 can then be rewritten as follows.

2.12 $$\Delta F \cong T(\tan\theta' - \tan\theta)$$

An interesting point to note is that $\tan\theta$ and $\tan\theta'$ represent a small change in the f direction (string position) over a small change in the z direction. Therefore, equation 2.12 can be rewritten as

2.13 $$\Delta F \cong T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

Since the middle part of equation 2.13 represents the change in the change over a distance of Δz, it can be rewritten as a second order partial derivative. Newton's second law also states

2.14 $$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

where μ is mass per unit length. This is because the second order partial derivative represents acceleration and its constant multiplier represents mass (the mass per unit length multiplied by unit length produces mass.) Equations 2.13 and 2.14 can now be set equal to each other to produce

2.15 $$T\frac{\partial^2 f}{\partial z^2}\Delta z = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

2.16 $$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}$$

Equation 2.16 is a simplified version of equation 2.15. Equation 2.16 can be rewritten in a better fashion since

2.17 $$v = \sqrt{\frac{T}{\mu}}$$

where v is the speed of propagation. The new version of equation 2.16 is

2.18 $$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

This is the electromagnetic wave equation. The general solution is

2.19 $$f(z,t) = g(z - vt) + h(z + vt)$$

This is because all linear combinations of independent solutions are represented in this solution, making it the most general form. Keep in mind that the function

2.20 $$f(z,t) = h(z + vt)$$

was added because the wave can be propagating in the negative direction. Therefore, all independent solutions must satisfy the property of the function g or function h. Thus far, imaginary numbers are not necessary in the solution. This is because the differential equation is just a relation between two different second order derivatives. Therefore the functions $A_1 \sin\theta$ and $A_2 \cos\theta$, which are real, would satisfy the differential equation and would be solutions. The A_1 and A_2 values are used as arbitrary constants. The most common solution is

2.21 $$f(z,t) = A\cos[k(z - vt) + \delta]$$

In equation 2.21, A represents the amplitude of the wave, δ represents the phase constant. Equation 2.21 can be rewritten in terms of ω, the angular velocity, to account for properties of a wave.

2.22 $$\omega = 2\pi\nu = \frac{2\pi}{T} = kv \qquad T = \frac{2\pi}{kv}$$

$\therefore$

2.23 $$f(z,t) = A\cos(kz - \omega t + \delta)$$

Notice that there is no imaginary numbers in the derivation of the wave solution or in the solution itself; it appears to be a notational convenance in this circumstance. Equation 2.1 is not necessary in describing electromagnetic waves, it can purely be described using equation 2.23. Equation 2.23 is equal to equation 2.1 using basic complex analysis. Using Euler's formula

1.9 $$e^{i\theta} = \cos\theta + i\sin\theta$$

an operator can be created to take the real part or the imaginary part out.

2.24 $$\mathrm{Re}[A(e^{i\theta})] = \mathrm{Re}[A(\cos\theta + i\sin\theta)] = A\cos\theta$$

2.25 $$\mathrm{Im}[A(e^{i\theta})] = \mathrm{Im}[A(\cos\theta + i\sin\theta)] = A\sin\theta$$

∴

2.26 $$f(z,t) = A\cos(kz - \omega t + \delta) = \text{Re}[Ae^{i(kz-\omega t+\delta)}]$$

2.1 $$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)} \quad \tilde{A} \equiv Ae^{i\delta}$$

2.27 $$f(z,t) = \text{Re}[\tilde{f}(z,t)]$$

The notational trick may fool people because physicists rarely use function 2.23 and therefore describe waves as 2.1 instead, assuming their audience knows equation 2.27.

The notational convenience is used because of the differences between the exponential functions and trigonometric functions in their nature. When working with trigonometric functions, mathematicians and physicists constantly have to use multiple trigonometric identities to arrive at their specific conclusion. However, exponential functions are easier to work with because they eliminate the necessity of trigonometric identities. For example, one can apply transformations to exponential functions through multiplication (whereas multiplication represents a rotation in this context.) Therefore, complex numbers can be seen as essential to mathematics and physics when used in this type of substitution. This is because the essence of exponential functions within the complex plane allow one to encapsulate all the relations inherent in tiresome trigonometric functions.

Analysis of Quantum Mechanics

Schrödinger equation.

The most fundamental differential equation of quantum mechanics, the Schrödinger equation, founded by Erwin Schrödinger, a twentieth century physicist, contains an imaginary number.

3.1 $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x,t)\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

Since the equation produces a solution used to find a probability distribution of the location of a particle, imaginary numbers appear in yet another physical context. The derivation of this equation will be analyzed, in order to track the significance of the imaginary number.

The following derivation is guided by Robert Eisberg, a professor of physics at UC Santa Barbara and Robert Resnick's, a professor of physics at Rensselaer Polytechnic Institute, proof; however, the derivation in this investigation takes a slightly different approach [2].

The Schrödinger equation relied on a few fundamental experiments, including the photoelectric effect and the electron crystal diffraction experiment. Therefore, these experiments should be discussed to see how they set up the parameters for the Schrödinger equation. The de Broglie postulate provided the motivation for the electron crystal diffraction experiment; it derived wave-type relations from relativistic principles that was supposed to describe aspects of particles. The electron crystal diffraction experiment demonstrates this wave-type aspect of particles. This experiment shot electrons into a crystal structure (a structure with a uniform nature) at different angles. It found that the electrons did not come back out of the structure at particular angles. It became evident that the particles have some wave-type property in the sense that the particle interferes with itself. In other words, it was shown that the particle was a wave in this context in which the matter constructively and destructively interferes with itself. The de Broglie postulate, which will be shown in the derivation of the Schrödinger equation, was proven to be true using this experiment. The photoelectric effect shows how light, which was already known to have wave-type properties, also exhibits particle-type properties. This experiment was designed to track how much energy was needed to ionize an electron from a metal plate using light. It experiment found that the electron was not continuously given energy, as one would expect a wave to behave; instead, the energy of the light was quantized into discrete particles known as photons. Since the photons have an extremely low chance of hitting an electron twice, each photon had to exceed a certain frequency in order have enough energy to ionize the electron.

At this point, Schrödinger wants to develop an equation to model the energy of a particle using its wave-type relations discussed above. As a result, he associated the values in the de Broglie postulate and the photoelectric effect with classical mechanical equations. The following relations are from the de Broglie postulate and the quantization of energy.

3.2 $$\lambda = \frac{h}{p}, \quad E_{tot} = h\nu$$

Then, kinetic energy is formulated in terms of momentum.

3.3 $$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} \quad p = mv$$

3.4 $$TE = \frac{p^2}{2m} + V(x,t), \quad \mathrm{PE} = V$$

Thereby allowing kinetic energy and total energy to be written in terms of microscopic attributes.

3.5 $$\frac{h^2}{2m\lambda^2} + V(x,t) = hv$$

which is in the form

3.6 $$\mathrm{KE} + \mathrm{PE} = \mathrm{TE}$$

In the case of the free particle, the potential energy will remain constant

3.7 $$V(x,t) = V_0$$

∴

3.8 $$F = 0$$

since

3.9 $$F = -\frac{\partial V(x,t)}{\partial x}$$

From here, the goal is to create a differential equation that describes this relationship at the quantum level. It is a reasonable assumption to conclude that the desired function will be sinusoidal in order to account for the wave-type properties that define the energy equation. In order to simplify the energy equation, new constants can be introduced to represent a variety of constants.

3.10 $$k = \frac{2\pi}{\lambda} \quad \omega = 2\pi v$$

3.11 $$\frac{\hbar^2 k^2}{2m} + V(x,t) = \hbar\omega, \quad \hbar \equiv \frac{h}{2\pi}$$

From here, a differential equation can be created so that, when the function for the de Broglie wave is plugged in, it produces equation 3.11. In order to accomplish this, it is necessary to examine some properties of the solution. The solution has to depend on *x* and *t* since the energy equation does. Since the derivative of a normalized wave produces the exact same equation with constants multiplied to the front due to chain rule, it is safe to assume that there will be a second order derivative on the left and a first order derivative on the right. As a result, each will be needed to be with respect to different variables since there are different constants on each side of the equation.

3.12 $$A_1\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x}+?V(x,t)=A_2\hbar\frac{\partial\psi(x,t)}{\partial t}$$

From equation 3.12, it can be assumed that ψ will have the following input

3.13 $$(kx-\omega t)$$

One interesting thing to realize is that the relation is equating a second order derivative to a first order derivative; real waves do not work this way. The $\sin\theta$ and $\cos\theta$ functions alternate every derivative; therefore, a first derivative cannot be equated to a second derivative. The next possibility is to consider a combination of the functions $\sin\theta$ and $\cos\theta$. This way, they would both outcomes are expressed. Furthermore, since the derivative of a wave always produces the same wave again (and chain rule components), there should be another wave equation multiplied by the potential function in order to cancel the other wave out and reproduce the energy equation. This wave should be the same as ψ itself in order to cancel everything out. The next step is to plug the hypothesized solution

3.14 $$\psi(x,t)=\alpha\sin(kx-\omega t)+\beta\cos(kx-\omega t)$$

into the differential equation that represents the energy of particles.

3.15 $$A_1\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x}+\psi(x,t)V(x,t)=A_2\hbar\frac{\partial\psi(x,t)}{\partial t}$$

Some may consider it questionable that coefficients are put in front of the wave functions since it should work for all real numbers if the differential equation is linear. However, it is important to entertain the possibility that the coefficients α and β might contain more than just constant values. Either way, it should not matter that these are written as all values should work since it is a linear differential equation. The following equations then follow as a result of this hypothesized solution.

3.16 $$\frac{\partial^2\psi}{\partial x} = -\alpha k^2 \sin(kx-\omega t) - \beta\cos(kx-\omega t)k^2$$

3.17 $$\frac{\partial\psi}{\partial t} = -\alpha\omega\cos(kx-\omega t) + \beta\omega\sin(kx-\omega t)$$

These can then be substituted into the differential equation.

3.18 $$\left(A_1\frac{\hbar^2}{2m}\right)[-k^2\alpha\sin(kx-\omega t) - k^2\beta\cos(kx-\omega t)] + \alpha V_0\sin(kx-\omega t) + \beta V_0\cos(kx-\omega t) - (A_2\hbar)[-\alpha\omega\cos(kx-\omega t) + \beta\omega\sin(kx-\omega t)] = 0$$

3.19 $$\left[-\frac{A_1\hbar^2k^2\alpha}{2m} + V_0\alpha + A_2\hbar\omega\beta\right]sin(kx-\omega t) + \left[-\frac{A_1\hbar^2k^2\beta}{2m} + V_0\beta - A_2\hbar\omega\alpha\right]\cos(kx-\omega t) = 0$$

The coefficients are then selected to make the left side zero since the right hand side is zero.

3.20 $$-\frac{A_1\hbar^2k^2\alpha}{2m} + V_0\alpha + A_2\hbar\omega\beta = 0$$

3.21 $$-\frac{A_1\hbar^2k^2\beta}{2m} + V_0\beta - A_2\hbar\omega\alpha = 0$$

Then the top equation can be divided by the bottom equation to obtain the following

3.22 $$\frac{\alpha}{\beta} = -\frac{\beta}{\alpha}$$

$\therefore$

3.23 $$\alpha^2 = -\beta^2 \quad \pm\sqrt{\alpha^2} = \pm\sqrt{-\beta^2} \quad \pm\alpha = \pm i\beta$$

This demonstrates that one of the constants must contain an imaginary number. In order to utilize Euler's formula, the coefficient of $\sin\theta$ is chosen to contain the imaginary number. And, of course, the simplest constants would be $+1$.

3.24 $$\psi(x,t) = i\sin(kx-\omega t) + \cos(kx-\omega t) = e^{i(kx-\omega t)}$$

The imaginary number is in the solution for the differential equation that models the energy for a particle. Although the the function cancels itself out to produce the energy equation, there is an imaginary number in the differential equation and its solution. Also, this wave function is involved in other physical applications. Notice how this proof does not reveal the physical significance of the imaginary number. This will be explored in further conceptual and philosophical questions as other applications of this wave function are analyzed.

Born's interpretation of the Schrödinger equation.

According to Eisberg and Resnick, Born, a twentieth century physicist, realized that the normalized solution to the Schrödinger equation, when multiplied by its complex conjugate, produces a probability distribution [2].

4.1 $$\rho = \psi \psi^*$$

This relation uses ψ to come to a conclusion that describes a physical system. However, the method by which this physical result is obtained involves cancelling out the imaginary number. Therefore, it is unclear on the extent to which the imaginary number corresponds to reality.

solution to the hydrogen atom.

Examining an actual solution to the Schrödinger equation will provide the full context of the Schrödinger equation and Born's probability distribution. The following derivation for the solution of the hydrogen atom is from Dr. Griffiths [4]. However, the explanations are not as in-depth.

5.1 $$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{\frac{-r}{na}} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right)\right] Y_l^m(\theta, \phi)$$

5.2 $$Y_l^m(\theta, \phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos\theta)$$

Due to the countless amount of computations and differential equation techniques needed to produce this solution, only the main parts of this example's derivation will be recreated and analyzed. Of course, the motivation here would be to discover its probability distribution for each state.

The following equations define the time-independent Schrödinger equation in three dimensional cartesian space

5.3 $$E\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

5.4 $$\int |\psi|^2 d^3\mathbf{r} = 1 \quad \mathbf{r} = (x, y, z)$$

with the general time-independent solution being

5.5 $$\psi(\mathbf{r}, t) = \psi_n(\mathbf{r}) e^{-i\frac{E_n t}{\hbar}}$$

The Schrödinger equation in spherical coordinates (which are used for atoms) is defined as

5.6 $$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial\phi^2}\right)\right] + V\psi = E\psi$$

The solution is assumed to be multiplicatively separable,

5.7 $$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

with each function having a dependance in only one of the three spherical dimensions. This is proven by converting the separable differential equations for each angular function to a recognizable function, notably, the associated Legendre function and the Legendre polynomial.

5.8 $$\Theta(\theta) = AP_l^m(\cos\theta)$$

5.9 $$P_l^m(x) \equiv [1-(\cos\theta)^2]^{\frac{|m|}{2}}\left(\frac{d}{d\theta}\right)^{|m|}P_l(x)$$

5.10 $$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{d\theta}\right)^l\left[(\cos\theta)^2-1\right]^l$$

5.11 $$\Phi(\phi) = e^{im\phi}$$

Notice that only the ϕ angle is in relation to the imaginary number. Furthermore, the total angular dependence is expressed by

5.12 $$Y_l^m(\theta,\phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta) \quad \epsilon = (-1)^m \text{ for } m \geq 0 \quad \epsilon = 1 \text{ for } m \leq 0$$

Since the angular function does not depend on the potential function. The next step is to determine the radial function.

5.13 $$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

5.14 $$\frac{1}{\kappa^2}\frac{\partial^2 u}{\partial r^2} = \left[1-\frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{(\kappa r)}+\frac{l(l+1)}{(\kappa r)^2}\right]u \quad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \quad u(r) = rR(r)$$

Then substitutions are made to simplify things.

5.15 $$\frac{d^2u}{d\rho^2} = \left[1-\frac{\rho_0}{\rho}+\frac{l(l+1)}{\rho^2}\right] \quad \rho \equiv \kappa r \quad \rho_0 \equiv \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}$$

After this, the asymptotic behavior is analyzed and then a solution is proposed in terms of another function.

5.16 $$u(\rho) = \rho^{l+1}e^{-\rho}\nu(\rho)$$

This new function is used to rewrite the differential equation, in which the solution can safely be assumed to be a power series.

5.17 $$\rho \frac{d^2\nu}{d\rho^2} + 2(l+1-\rho)\frac{d\nu}{d\rho} + [\rho_0 - 2(l+1)]\nu = 0$$

5.18 $$\nu(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

Then the power series is plugged into the differential equation in which coefficients are equated. After evaluating asymptotic behavior, it is found that the series must terminate. From the previous statements, it is found that

5.19 $$\nu(\rho) = c_0 e^{2\rho}$$

or

5.20 $$\nu(\rho) = L^{2l+1}_{n-l-1}(2\rho)$$

is true, where

5.21 $$L^{p}_{q-p}(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_q(x)$$

5.22 $$L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q (e^{-x}x^q)$$

Furthermore,

5.23 $$u(\rho) = c_0 \rho^{l+1} e^{\rho}$$

5.24 $$\rho_0 = 2n \quad n \equiv j_{\max} + l + 1$$

∴

5.25 $$E = -\frac{me^4}{8\pi^2\epsilon_0^2\hbar^2\rho_0^2} \quad E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2} \quad n = 1,2,3,...$$

The first energy state is -13.6eV, which means +13.6eV needs to be supplied in order to free the electron from its bound state. Any lower amount of energy will not ionize the electron. The radial function, which is not itself complex, but is in a complex function that has unclear physical relevance, has proven to identify different energy states. The only real part of $\psi$is the only necessary part for the energy states.

Normalizing the radial equation creates more known constants. This has to be done each time a particular ψ function is being solved. From the definitions of all the previous functions, the following can now be defined

5.26 $$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{\frac{-r}{na}} \left(\frac{2r}{na}\right)^l \left[L^{2l+1}_{n-l-1}\left(\frac{2r}{na}\right)\right] Y^m_l(\theta,\phi) \quad a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$$

5.27 $$\int \psi^*_{nlm}\psi_{n'l'm'} r^2 \sin\theta \, dr \, d\theta \, d\phi = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

de Broglie-Bohm pilot wave theory.

Although this theory is disregarded among most quantum mechanical physicists because of its philosophical views, it gives a less vague definition of the particle and ψ. According to Travis Norsen, a doctor of nuclear astrophysics, the particle is defined by the following relations [5].

6.1 $$\frac{dX}{dt} = \frac{j}{\rho}\bigg|_{x=X(t)}, \quad j = \frac{\hbar}{2mi}(\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*), \quad \rho = |\psi|^2$$

6.2 $$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

Notice how the velocity function is defined by a particle position function. Also note that this is the simple, one dimensional equation. The theory says that ψ is the "guiding function" of the particle. In other words, a physical attribute related to the ψ function somehow pushes the particle around. Therefore, the particle does not exist everywhere at once. ψ is given the physical relevance of physically pushing the particle around.

Discussion

Analyzing the one-dimensional electromagnetic wave equation demonstrated that imaginary numbers were not involved in its proof. Therefore, imaginary numbers are not needed to understand the physical relevance of electromagnetic waves. However, it was shown that calculations become easier if imaginary numbers are introduced because of the absence of trigonometric functions. Ultimately, the use of imaginary numbers is beneficial to electrodynamics due to the imaginary number's ability to condense information into an easier function (the exponential function.)

Analyzing the derivation of Schrödinger's equation has demonstrated that imaginary numbers are required, in order for the differential equation to produce the quantum energy equation that it was derived from. This is because the solution must always contain an imaginary number in its general form. However, the differential equation and its solution does not hold to the level of a proof. Since it is a plausibility argument, the only way the Schrödinger equation is connected to reality is that it yields correct results when used in its probabilistic interpretation and when the solution is plugged in, it gives an energy equation.

3.11 $$\frac{\hbar^2 k^2}{2m} + V(x,t) = \hbar\omega, \quad \hbar \equiv \frac{h}{2\pi}$$

The reason for this procedure, without knowing the physical implications of ψ, is unclear.. This is a result of purely deductive reasoning. The fact that ψ is complex-valued does not become an immediate problem since the function has no direct physical physical application. However, when viewed in conjunction with Born's probabilistic interpretation of ψ or with the de Broglie-Bohm pilot wave theory, imaginary numbers are connected to reality in a more distinct way.

This investigation has demonstrated that imaginary numbers are necessary for Schrödinger's equation, while reminding the audience that Born's physical, probabilistic interpretation is dependent on Schrödinger's equation. If both of these theorems are true and not some coincidence or some consequence of a deeper, unknown relation, then imaginary numbers are absolutely necessary to describe the probability distribution for the position of a particle. Therefore, there is a connection between reality and imaginary numbers. The connection is one such that a certain physical system can only be expressed in terms of a complex wave function and its complex conjugate. From the perspective of a researcher who views reality as objective, this may point out a flaw in either theorem. However, Born does not prescribe physical relevance to any part of his theorem.

Although the de Broglie-Bohm pilot wave theory does predict the same results, it is less vague than the general quantum mechanics theorems. Therefore, this theorem shows a correlation between imaginary numbers and reality because it prescribes physical relevance to the wave that pushes the particle which is dependent on ψ.

This paper did not analyze a lot of other theories that had imaginary numbers. As a result, a generalization about the connection between imaginary numbers and reality could not be created.

Conclusion

Quantum theories need to be reevaluated; this investigation has shown that imaginary numbers are necessary for producing physical results. In fact, an interesting hypothesis to consider is that current quantum theories cannot rely on imaginary numbers because the theories are incomplete and, as a result, vague on the physical relevance of the wave function. These quantum theories may be a derivative (subset) of the real concept. In order to evaluate different theorems or to expand new ones, a new direction/perspective may be needed. Since the pilot-wave theory used a more philosophically correct approach, perhaps a new, expanded on pilot-wave theory could be produced that does not rely on imaginary numbers. Quantum theories needing to be reevaluated also relates to the fact that these theories cannot be related to the other fields of physics.

These theories were arrived at not only through an incorrect philosophical method, but through serious deduction that was unmotivated by physical observations. Mathematics and physics cannot have as many breakthroughs with these fundamentally wrong methodologies, which leads to less technological innovation and a lesser quality of life.

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